Mathematical Induction

Solutions

1.
Test Case:

For n = 1, the left hand side and right hand side both equal 1

Then, suppose that $1+2+...+k=\dfrac{k(k+1)}{2}$ is true.

Induction Step:

$$1+2+...+k+(k+1)$$

$$=\frac{k(k+1)}{2}+(k+1)$$

$$=\frac{k(k+1)}{2}+\frac{2(k+1)}{2}$$

$$=\frac{(k+2)(k+1)}{2}$$

$$=\frac{[(k+1)+1](k+1)}{2}$$

If the statement is true for n = k, it must be true for n = k+1, thus the statement is true for all integers.

2.

Test Case:

For n = 1, the left hand side and right hand sides both equal 1.

Now, suppose that

$$1^2 + 2^2 + ... + k^2 = \frac{k(k+1)(2k+1)}{6}$$

is true

Induction Step:

$$1^2 + 2^2 + ... + k^2 + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}$$

$$= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6}$$

$$= \frac{(k+1)(2k^2 + k + 6k + 6)}{6}$$

$$= \frac{(k+1)(2k^2 + 7k + 6)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$$

If the statement is true for n = k, it must be true for n = k+1, thus the statement is true for all integers.

3.

Test Case:

For n = 1, $n(n+1)(n+2)=6=3(2)$

Now, assume that $k(k+1)(k+2)$ is a multiple of 3

Induction Step:

$$(n+1)(n+2)(n+3)$$

$$=n(n+1)(n+2)+3(n+1)(n+2)$$

The first term is a multiple of 3 by our assumption and the second is a multiple of 3 because of the 3 in front of it.

If the statement is true for n = k, it must be true for n = k+1, thus the statement is true for all integers.

4.

Test Case:

For n = 1, both the left hand side and right hand side equal 2.

Now, assume

$$2+4+...+2(k-1)+2k=k^2+k$$

Induction Step:

$$2+4+6+...+2(k-1)+2k+2(k+1)$$

$$=k^2+k+2(k+1)$$

$$=k^2+3k+2$$

$$=(k+1)^2+(k+1)$$

If the statement is true for n = k, it must be true for n = k+1, thus the statement is true for all integers.

5.

Test Case:
For n = 1, both the left hand side and right hand side equal 1

Now, assume

$$1+3+5+7+...+(2k-1)=k^2$$

Induction Step:

$$1+3+5+...+(2k-1)+(2k+1)$$

$$=k^2+2k+1$$

$$=(k+1)^2$$

If the statement is true for n = k, it must be true for n = k+1, thus the statement is true for all integers.

6.
Test Case:
For n = 1, both the left hand side and right hand side equal 1

Now, assume

$$1+4+7+10+...+(3k-2)=\frac{k}{2}(3k-1)$$

Induction Step:

$$1+4+\ldots+(3k-2)+[3(k+1)-2]$$

$$=\frac{k}{2}(3k-1)+(3k+1)$$

$$=\frac{3k^2}{2}-\frac{k}{2}+3k+1$$

$$=\frac{3k^2}{2}+\frac{5k}{2}+1$$

$$=\frac{3k^2+5k+2}{2}$$

$$=\frac{3k^2+3k+2k+2}{2}$$

$$=\frac{3k(k+1)+2(k+1)}{2}$$

$$=\frac{(k+1)}{2}(3k+2)$$

$$=\frac{(k+1)}{2}[3(k+1)-1]$$

If the statement is true for n = k, it must be true for n = k+1, thus the statement is true for all integers.

7.
Test Case:
For n = 1, both sides equal 2, so the inequality is true.

Assume $2^k \geq 2k$ is true.

Induction Step:

$$2^k \geq 2k$$

$$2^{k+1} = 2 \times 2^k \geq 2 \times 2k = 4k \geq 2(k+1)$$

Why is $4k \geq 2(k+1)$?

Because, we're only looking at $k \geq 1$. And if we solve the inequality, we find it works for the interval we're looking at:

$$4k \geq 2(k+1)$$

$$4k \geq 2k+2$$

$$2k \geq 2$$

$$k \geq 1$$

If the statement is true for n = k, it must be true for n = k+1, thus the statement is true for all integers.

8.
Test Case:
For n = 1, the left side and right side both equal 1

Assume $1+3+9+27+...+3^{k-1}=\frac{3^k-1}{2}$

Induction Step:

$$1+3+9+27+...+3^{k-1}+3^k$$

$$=\frac{3^k-1}{2}+3^k$$

$$=\frac{3^k-1}{2}+\frac{2\times 3^k}{2}$$

$$=\frac{3\times 3^k-1}{2}$$

$$=\frac{3^{k+1}-1}{2}$$

If the statement is true for n = k, it must be true for n = k+1, thus the statement is true for all integers.

9.
Test Case:

Since the statement requires $n\geq 10$, our test case will be for n = 10. In this case on the left, we have 1024 and on the right we have 1000. So the statement is true.

Now assume $2^k \geq 100k$ is true.

Induction Step:

$$2^{k+1} = 2 \times 2^k \geq 2 \times 100k = 200k \geq 100(k+1)$$

If the statement is true for n = k, it must be true for n = k+1, thus the statement is true for all integers.

In the last step, we used the fact that $200k \geq 100(k+1)$ when $k \geq 1$.

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